Improper Integral

1. State whether the following converge or diverge. Evaluate, if the integral converges:

(a) $\int_1^{\infty} \frac{dx}{x\sqrt{2x^2-1}}$ **(b)** $\int_{-\infty}^{\pi/4} \sin 2x\,dx$ **(c)** $\int_{-\infty}^{-1} \frac{2}{x^2}dx$ **(d)** $\int_{-\infty}^{\infty} \frac{dx}{x^2(1+x^2)}$

(e) $\int_0^{\infty} xe^{-x^2}dx$ **(f)** $\int_{-1}^{\infty} \frac{dx}{(x+2)^{3/2}}$ **(g)** $\int_{-1}^{\infty} \frac{dx}{x+2}$ **(h)** $\int_0^{\infty} \frac{dx}{4x^2+25}$

(i) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ **(j)** $\int_1^{a} \frac{dx}{(x-1)^2}$ **(k)** $\int_{-\infty}^{\infty} \frac{dx}{a^2+b^2x^2}$, a, b > 0 **(l)** $\int_0^{\infty} e^{-x}dx$

(m) $\int_0^{\infty} e^{-x}\sin x\,dx$ **(n)** $\int_0^{1} \ln x\,dx$ **(o)** $\int_0^{a} \frac{x\,dx}{\sqrt{a^2-x^2}}$, a > 0 **(p)** $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

(q) $\int_1^{\infty} \frac{dx}{x(1+x)}$ **(r)** $\int_1^{\infty} \frac{dx}{x^2(1+x^2)}$

2. Let $I_n = \int_0^{\infty} x^n e^{-x}dx$. Show that $I_n = n\,I_{n-1}$. Hence evaluate I_n , where $n \in \mathbf{N}$.

3. Evaluate, by reduction formula, or otherwise, where $n \in \mathbb{N}$:

(a) $\int_{-\infty}^{\infty} \frac{dx}{(ax^2+2bx+c)^n}$ $(ac - b^2 > 0,\ a > 0)$ **(b)** $\int_1^{\infty} \frac{dx}{x(x+1)\ldots(x+n)}$ **(c)** $\int_0^{1} \frac{x^n dx}{\sqrt{1-x^2}}$

4. Prove that **(i)** $\int_0^{\pi/2} \ln(\sin x)dx = \int_0^{\pi/2} \ln(\cos x)dx = \frac{1}{2}\int_0^{\pi/2} \ln(\sin 2x)dx - \frac{\pi}{4}\ln 2$;

(ii) $\int_0^{\pi/2} \ln(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \ln(\sin x)dx = \int_0^{\pi/2} \ln(\sin x)dx$

Deduce that $\int_0^{\pi/2} \ln(\sin x)dx = \frac{\pi}{2}\ln\frac{1}{2}$.

5. Obtain a reduction formula for $\int \cos^{2n} x\,dx$, and evaluate $\int_0^{r\pi/2} \cos^{2n} x\,dx$, where $r \in \mathbf{N}$.

6. **(i)** Evaluate $\int e^{-x^2}x^3dx$ and $\int_0^{x} \frac{dx}{(1+x^2)^3}$.

(ii) Show that $\int_0^{\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{4}$. Hence, or otherwise, evaluate $\int_0^{\pi} \frac{(\cos\theta+2\sin\theta)d\theta}{5+3\cos\theta}$.

7. Integrate with respect to x : **(i)** $\frac{4-x}{\sqrt{3+2x-x^2}}$ **(ii)** $\frac{\ln(1+x)}{x^2}$

8. Show that $\int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx$ and prove that $\int_0^{\pi} \frac{x\,dx}{4+\sin^2 x} = \pi^2\frac{\sqrt{5}}{20}$.

9. If $I_n = \int \sec^n\theta\,d\theta$, show that , when $n \ge 1$, $(n-1)\,I_n = \sec^{n-2}\theta\tan\theta + (n-2)\,I_{n-2}$.

Show that $8\int_0^{\pi/4} \sec^5\theta\,d\theta = 7\sqrt{2} + 3\ln(1+\sqrt{2})$.

10. If $I_{m,n} = \int_0^{\pi/2} \cos^m\theta\sin^n\theta\,d\theta$, where $m, n \in \mathbf{N}$ with $m > 1$, find $I_{m,n}$ in terms of $I_{m-2,n}$.

Evaluate $\int_0^{\infty} \frac{t^2dt}{(1+t^2)^4}$.